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On Quadruple Theta-Functions.

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Part I.

The following paper is intended to be simply introductory to the theory of the theta-functions of four variables. I have followed the method adopted by Prof. Cayley in his well-known memoir on the double theta-functions; the earlier pages of the present Part being indeed simply the extension of the work already done by Prof. Cayley for the theta-functions with two arguments. In the present Part I give the elementary theorems connected with the quadruple theta-functions and a product table for the "square-products," *i. e.* the products of two functions having the same characteristics but not the same arguments. In Part II I intend to go more fully into the theory of the characteristics and to develop the relations existing between different groups of quadruple theta-functions, that is, the relations similar to the Göpel and Kummer relations connecting the products and squares of the double theta-functions. The only paper that I am aware of which directly treats of the quadruple functions is one by Nöther mentioned below. In this paper Nöther deals almost entirely with certain properties of the characteristics. It is known (the proof for completeness is given below) that every characteristic with the exception of $\begin{pmatrix}0000\\0000\end{pmatrix}$ can be divided in 128 different ways into the sum of two, viz. in 64 ways into the sum of an odd and of an even characteristic; in 36 different ways into the sum of two even characteristics, and in 28 different ways into the sum of two odd characteristics. Denoting any characteristic by (h) and in particular the characteristic $\begin{pmatrix}0000\\0000\end{pmatrix}$ by (0), the system of 28 pairs of odd characteristics in which each characteristic (h), excepting (0), can be divided is called a *group,* and this group Nother calls a group-characteristic and denotes by $[\alpha]$. If

$$[\alpha] = (a_1) + (a_2) = (a'_1) + (a'_2) = \text{etc.}, \ldots$$

where (a_1), (a_2), (a'_1), (a'_2), etc., are all odd, Nöther says that (a_1), (a_2), (a'_1), etc., are *contained* in the group $[\alpha]$, and further that (a_1) and (a_2), etc., are *paired*—

giving thus in all 255 pairs. Nöther has here extended Weber's investigation on the triple theta-functions, particularly that part of the latter's work which refers to what he calls "*complete*" 7-*systems*; Nöther shows that there are for the quadruple functions certain analogous 8-systems, that is, systems of 8 odd characteristics for which the sum of any 5 is odd, and the sum of any 3 or of any 7 is even. In the last section of his paper he gives as an application of the relations existing between these theta-functions with zero arguments, the reduction to the least number of conditions which must hold in order that the theta-functions may become hyperelliptic functions, showing finally that this only requires the vanishing of *three* even theta-functions. In the following list of memoirs I have enumerated only those with which I am acquainted and know to have a direct bearing upon the subject of the quadruple functions. Although none but the one already mentioned deal directly with these functions, all refer to them more or less directly.

NÖTHER. Zur Theorie der Thetafunctionen von vier Argumenten: *Mathematische Annalen*, xiv, pp. 249–293.

—— Zur Theorie der Thetafunctionen von beliebig vielen Argumenten: *Mathematische Annalen*, xvi, pp. 270–344.

PRINGSHEIM. Zur Theorie der hyperelliptischen Functionen, insbesondere derjenigen dritter Ordnung ($\rho = 4$): *Mathematische Annalen*, xii, pp. 435–475.

STAHL. Das Additionstheorem der ϑ-Functionen mit p Argumenten: *Jour. für die reine und ange. Math.*, 88, pp. 117–130.

—— Beweis eines Satzes von *Riemann* über ϑ-characteristiken: *Jour. für die reine und ange. Math.*, 88, pp. 273–276.

FROBENIUS. Zur Theorie der Transformation der Thetafunctionen: *Jour. für die reine und ange. Math.*, 89, pp. 40–46.

—— Ueber das Additionstheorem der Thetafunctionen mehrerer Variabeln: *Jour. für die reine und ange. Math.*, 89, pp. 185–220.

BRIOSCHI. La relazione di Göpel per funzioni iperellittiche d'ordini qualunque: *Annali di Mat.*, Ser. II^a, Tomo X°, pp. 161–172.

WEIERSTRASS. Zur Theorie der Abel'schen Functionen: *Jour. für die reine und ange. Math.*, Vols. 47 and 52.

WEBER. Ueber die Transformationstheorie der Thetafunctionen: *Annali di Mat.*, Ser. II^a, Tomo IX°.

PRYM. Untersuchungen über die Riemann'sche Thetaformel und die Riemann'sche charakteristikentheorie: Leipzig, 1882.

Weber's paper I have never seen, but have come across numerous references to it; Prym's paper is published separately, and is a most valuable one for the student interested in the subject of theta-functions. Frobenius, in the second of his papers above mentioned, takes as his point of departure a certain result arrived at by Weber and subsequently specialized in a most important way by Hermite. Weber shows that the transformed theta-functions of ρ variables can be expressed as integral functions of the k^{th} order of the original $2^{2\rho}$ theta-functions, k denoting the order of the transformation. Hermite shows for the case $\rho = 2$, that if ϑ_0, ϑ_1, ϑ_2, ϑ_3 denote four of the double ϑ-functions connected by the Göpel bi-quadratic relation, and if Θ_0, Θ_1, Θ_2, Θ_4 denote the transformed functions, the four functions Θ_a can be expressed as integral functions of the four corresponding functions ϑ_a. Frobenius discusses then the question as to whether or not a similar property can be shown to exist for the theta-functions of more than two arguments, and gives application of his results to the cases of $\rho = 2$, 3 and 4. The subject of the transformation of the quadruple functions is reserved for the second part of the present paper. Before leaving this introductory section, however, reference should be made to two papers by Clifford, both contained in his "Mathematical Papers," viz., "On groups of periodic functions," which deals with multiple theta-functions and follows Rosenhain's method; and "Theory of marks of multiple theta-functions," in which Clifford partly follows the general methods of Riemann and partly the particular methods employed by Weber in his memoir "Theorie der Abelschen Functionen vom Geschlecht 3."

The theta-function of four variables depends upon 10 parameters,

$$(a_{11} a_{12} \ldots a_{44})$$

which are the coefficients of a quadric function of four ultimately disappearing integers, say m_1, m_2, m_3, m_4, viz.

$$(a_{11} a_{12} \ldots a_{44} \between m_1, \ldots m_4)^2;$$

upon four arguments u_1, u_2, u_3, u_4, and upon 8 characteristics α_i and β_i, $i = 1, 2, 3, 4$, which are each either 0 or 1. The arrangement

$$\begin{pmatrix} \alpha_1 \, \alpha_2 \, \alpha_3 \, \alpha_4 \\ \beta_1 \beta_2 \beta_3 \beta_4 \end{pmatrix}$$

is called a *characteristic*, or, in Clifford's notation, a *mark*. It is easy to see for α and β each equal 0 or 1 that we have in all $2^{2 \times 4} = 256$ different characteristics, and consequently 256 quadruple theta-functions.

The integers m_i will be assumed all even, and we write

$$\begin{pmatrix} m_1 m_2 m_3 m_4 \\ u_1\ u_2\ u_3\ u_4 \end{pmatrix} = \frac{1}{4}(a_{11}a_{12}\ldots a_{44}\between m_1 m_2 m_3 m_4)^2 + \frac{1}{2}\pi i(m_1u_1 + m_2u_2 + m_3u_3 + m_4u_4)$$

and equally

$$\begin{pmatrix} m_1+\alpha_1, & m_2+\alpha_2, & m_3+\alpha_3, & m_4+\alpha_4 \\ u_1+\beta_1, & u_2+\beta_2, & u_3+\beta_3, & u_4+\beta_4 \end{pmatrix}$$
$$= \frac{1}{4}(a_{11}a_{12}\ldots a_{44}\between m_1+\alpha_1\ldots m_4+\alpha_4)^2$$
$$+\frac{1}{2}\pi i((m_1+\alpha_1)(u_1+\beta_1)+\ldots+(m_4+\alpha_4)(u_4+\beta_4)).$$

The quadruple theta-functions are now defined by the equation

$$\vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(u_1u_2u_3u_4) = \sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}\exp.\begin{pmatrix} m_1+\alpha_1, & m_2+\alpha_2, & m_3+\alpha_3, & m_4+\alpha_4 \\ u_1+\beta_1, & u_2+\beta_2, & u_3+\beta_3, & u_4+\beta_4 \end{pmatrix}.$$

The summations extend over all positive and negative even integer values of $m_1\ldots m_4$. For brevity this may be written simply as

$$\vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(u) = \Sigma\exp.\begin{pmatrix}m+\alpha\\ u+\beta\end{pmatrix}.$$

It is well known that in order that the series here written may be convergent, the parameters a_{ik} must either be real, or if they are imaginary, that their real parts must be negative. The quantities α and β are all either zero or unity; it may happen that other values will appear for these quantities, but in each case, as will be seen, the odd values may be replaced by unity and the even values by zero. If $\Sigma\alpha\beta$ is even, the characteristic $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ is said to be even; if $\Sigma\alpha\beta$ is odd, the characteristic is said to be odd. The number of odd and of even characteristics is readily computed, and it is as easily done for p-tuple as for quadruple functions. Let O_{p-1} and E_{p-1} denote the number of odd and of even characteristics for a $p-1$-tuple function; then clearly $O_{p-1}+E_{p-1}=2^{2(p-1)}$. Prefix to each of the O_{p-1} odd characteristics $\begin{matrix}0\\0\end{matrix},\ \begin{matrix}0\\1\end{matrix},\ \begin{matrix}1\\0\end{matrix}$ and to each of the E_{p-1} even characteristics $\begin{matrix}1\\1\end{matrix}$ and so get an odd characteristic for the p-tuple function. We have then

$$O_p = 3\,O_{p-1}+E_{p-1} = 2^{2(p-1)}+2\,O_{p-1}$$
$$= 2^{2(p-1)} + 2.2^{2(p-2)} + 2^2.2^{2(p-3)} + \ldots + 2^{p-1} = 2^{p-1}(2^p-1)$$

and this for $p=4$ is $O_p=120$. For the number of even marks we have

$$E_p = 2^{2p} - O_p = 2^{p-1}(2^p+1)$$

or for $p=4$, $E_p=136$.

Given two characteristics $\binom{\alpha}{\beta}$ and $\binom{a}{b}$, their sum, or difference, is given by the symbol $\binom{\alpha \pm a}{\beta \pm b}$, *i. e.* for the case of 4 arguments

$$\begin{pmatrix}\alpha_1, & \alpha_2, & \alpha_3, & \alpha_4\\ \beta_1, & \beta_2, & \beta_3, & \beta_4\end{pmatrix} \pm \begin{pmatrix}a_1, & a_2, & a_3, & a_4\\ b_1, & b_2, & b_3, & b_4\end{pmatrix} = \begin{pmatrix}\alpha_1 \pm a_1, & \alpha_2 \pm a_2, & \alpha_3 \pm a_3, & \alpha_4 \pm a_4\\ \beta_1 \pm b_1, & \beta_2 \pm b_2, & \beta_3 \pm b_3, & \beta_4 \pm b_4\end{pmatrix}.$$

Denoting by $\binom{\alpha}{\beta}$ any characteristic, we may obviously write

$$\binom{\alpha}{\beta} = \binom{a}{b} + \binom{c}{d}$$

and either of the characteristics on the right-hand side of the equation may be arbitrarily assumed—giving, say for the other

$$\binom{c}{d} = \binom{\alpha}{\beta} - \binom{a}{b}.$$

Give now to $\binom{a}{b}$ all of the 2^{2p} possible values which it may have for a p-tuple theta-function, and we have a corresponding set of 2^{2p} values for $\binom{c}{d}$; but as it is quite clear that $\binom{a}{b} + \binom{c}{d} = \binom{c}{d} + \binom{a}{b}$ there are only $2^{2p} \div 2$ different divisors of $\binom{\alpha}{\beta}$, or, in other words, any characteristic $\binom{\alpha}{\beta}$ of a p-tuple theta-function can be divided in 2^{2p-1} different ways into the sum of two characteristics. For $p=4$ we have as above stated 128 different ways of dividing the characteristic of a quadruple function into the sum of two characteristics. Each of the possible decompositions of a given characteristic $\binom{\alpha}{\beta}$ into the sum of two others $\binom{a}{b}$ and $\binom{c}{d}$ may take place in one of three different ways: first, both $\binom{a}{b}$ and $\binom{c}{d}$ may be even; second, both may be odd, and third, one may be even and the other odd. Denote the number of the cases where $\binom{a}{b}$ and $\binom{c}{d}$ are both even by ξ, the number when both are odd by η, and when one is even and the other odd by ζ, required to find the values of ξ, η and ζ for a p-tuple theta-function.

The following method of obtaining the values of these quantities is essentially the same as that given by Prym—which again is almost identical with Riemann's process. It is necessary first, however, to prove a subsidiary theorem. Suppose we have a given characteristic $\binom{\alpha}{\beta}$ *i. e.* $\begin{pmatrix}\alpha_1\alpha_2 \dots \alpha_p\\ \beta_1\beta_2 \dots \beta_p\end{pmatrix}$; this can be divided into the sum of two others $\binom{a}{b}$ and $\binom{c}{d}$; assume either one of these, say $\binom{c}{d}$ as any one of the 2^{2p} characteristics of a p-tuple function. Write

$$H = \sum_{\binom{c}{d}} (-)^{\Sigma(\alpha_i d_i + \beta_i c_i)}$$

in which i takes all values from 1 up to p, and where the larger Σ refers to all possible values of $\binom{c}{d}$. Each c and each d of course take only one of the other of the values 0 and 1. Suppose now either c_i or d_i to alter by unity, that is, let either or both of them become $c_i + 1$ and $d_i + 1$, then the order of the summation under the small Σ is changed, but the value of the entire sum is unaltered; on the other hand, however, if c_i is increased by unity, H alters by the factor $(-)^{\beta_i}$, and if d_i is altered by unity, H takes the factor $(-)^{\alpha_i}$. Each of these factors being independent of $\binom{c}{d}$ can be placed before the large Σ. We have then $$H = (-)^{\alpha_i} H, \; H = (-)^{\beta_i} H,$$
that is $H = 0$, whenever of any one of the $2p$ quantities α_i and β_i (all of which are of course either 0 or 1) one at least is $= 1$; this excludes of course only the characteristic $\binom{0}{0}$. Now writing $\binom{\alpha}{\beta} = \binom{a}{b} + \binom{c}{d}$, where $\binom{\alpha}{\beta}$ is a given characteristic, write in place of $\binom{c}{d}$ all of the E_p even characteristics, and determine each corresponding $\binom{a}{b}$ by means of the equation $\binom{a}{b} = \binom{\alpha}{\beta} - \binom{c}{d}$.

It is clear then that in the decomposition of the characteristic $\binom{\alpha}{\beta}$ each of the ξ-divisions will come in twice and each of the ζ-divisions will come in once. Again give to $\binom{c}{d}$ all of the O_p odd values and we have that each of the η-divisions comes in twice and each of the ζ-divisions comes in once. Expressed algebraically these two statements are

$$E_p = 2\xi + \zeta, \quad O_p = 2\eta + \zeta.$$

Form now the expression $$(-)^{\Sigma c_i d_i} . (-)^{\Sigma a_i b_i}$$
and denote this by ϕ. It is obvious that ϕ takes the value $+1$ for either the ξ- or the η-divisions, and the value -1 for the ζ-divisions. Now since $\binom{a}{b}$ is known when $\binom{c}{d}$ is given, give to $\binom{c}{d}$ all its possible 2^{2p} values and take the

sum of the corresponding values of ϕ; call this sum σ; then since each possible division of $\binom{\alpha}{\beta}$ comes in twice, we have

$$\sigma = 2(\xi + \eta - \zeta).$$

It is quite clear that we can write

$$\phi = (-)^{\Sigma c_i d_i}.(-)^{\Sigma(\alpha_i + c_i)(\beta_i + d_i)} = (-)^{\Sigma\alpha_i\beta_i}.(-)^{\Sigma(\alpha_i d_i + \beta_i c_i)}.$$

Taking now the sum of all values of ϕ and we have, since as above shown

$$\sum_{\binom{c}{d}} (-)^{\Sigma(\alpha_i d_i + \beta_i c_i)} = 0$$

when the case $\binom{\alpha}{\beta} = \binom{0}{0}$ is excluded,

$$\sigma = \sum_{\binom{c}{d}} \phi = (-)^{\Sigma\alpha_i\beta_i}.\sum_{\binom{c}{d}} (-)^{\Sigma(\alpha_i d_i + \beta_i c_i)} = 0,$$

or finally
$$\xi + \eta - \zeta = 0.$$

Grouping together now the three equations obtained, viz.

$$E_p = 2\xi + \zeta \qquad O_p = 2\eta + \zeta \qquad 0 = \xi + \eta - \zeta$$

we have

$$\begin{aligned} 3\xi + \eta &= E_p = 3E_{p-1} + O_{p-1} \\ 3\eta + \xi &= O_p = 3\,O_{p-1} + E_{p-1} \\ \zeta &= \xi + \eta \end{aligned}$$

and finally

$$\begin{aligned} \xi &= E_{p-1} = 2^{p-2}(2^{p-1} + 1) \\ \eta &= O_{p-1} = 2^{p-2}(2^{p-1} - 1) \\ \zeta &= E_{p-1} + O_{p-1} = 2^{2(p-1)}. \end{aligned}$$

For $p = 4$ we have then for the number of divisions of a given characteristic $\binom{\alpha}{\beta}$ into the sum of two even characteristics $\binom{a}{b}$ and $\binom{c}{d}$ $\xi = 36$, into the sum of two odd $\eta = 28$, into the sum of one even and the other odd $\zeta = 64$, making in all $\xi + \eta + \zeta = 128$.

Leaving here for the present the consideration of the special properties of the characteristics, I now obtain some of the more elementary properties of the quadruple theta-functions. If each or any α be increased by an even integer there is no change in the value of the function. For suppose α to be replaced by $\alpha + 2x$ where x is an integer, then since m is even and takes all even values from $-\infty$ to $+\infty$ it is clear that $m + \alpha + 2x$ will take just the same values as $m + \alpha$ and so produce no effect on the function. If, however, the integers β be each or any of them increased by an even integer there is a change in the function. The quadratic term is obviously unaltered, but the linear term is easily seen to be increased by the quantity (y being an integer)

$$\pi i(m_1y_1+m_2y_2+m_3y_3+m_4y_4)+\pi i(\alpha_1y_1+\alpha_2y_2+\alpha_3y_3+\alpha_4y_4).$$

The first term of this is an even multiple of πi, say $=2n\pi i$; but $e^{2n\pi i}=1$, so that this term produces no effect. The second may be either an even or an odd multiple of πi, that is, may in the series give either the factor $+1$ or -1. We have then

$$\vartheta\binom{\alpha+2x}{\beta}(u)=\vartheta\binom{\alpha}{\beta}(u)$$

$$\vartheta\binom{\alpha}{\beta+2y}(u)=(-)^{\Sigma\beta y}\vartheta\binom{\alpha}{\beta}(u)$$

or combining these

$$\vartheta\binom{\alpha+2x}{\beta+2y}(u)=(-)^{\Sigma\beta y}\vartheta\binom{\alpha}{\beta}(u).$$

The only effect of altering the elements of the characteristic by odd integers is to interchange the functions. The even characteristics correspond to even theta-functions, and the odd characteristics to odd theta-functions, for remembering that m takes all (even) positive and negative values, it is obvious that $-m-\alpha$ takes precisely the same series of values as $m+\alpha$, so that in the function $\vartheta\binom{\alpha}{\beta}(-u)$ we may write the linear term as

$$\frac{1}{2}\pi i\Sigma(-m-\alpha)(-u+\beta)$$

or

$$\frac{1}{2}\pi i\Sigma(m+\alpha)(u+\beta)-\pi i\Sigma m\beta-\pi i\Sigma\alpha\beta.$$

Taking the exponential of this we have that the first term gives the linear part of the general term in the series; the second term gives the factor $+1$; the third term gives the factor $(-)^{\Sigma\alpha\beta}$, so that finally we have

$$\vartheta\binom{\alpha}{\beta}(-u)=(-)^{\Sigma\alpha\beta}\vartheta\binom{\alpha}{\beta}(u).$$

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Take an integer x, then we have obviously

$$\vartheta\binom{\alpha}{\beta}(u+x)=\vartheta\binom{\alpha}{\beta+x}(u).$$

so that when u is altered by an integer x the functions are interchanged. Now replace x by $2x$, *i. e.* alter u by an even integer, then we have

$$\vartheta\binom{\alpha}{\beta}(u+2x)=\vartheta\binom{\alpha}{\beta+2x}(u)=(-)^{\Sigma\alpha x}\vartheta\binom{\alpha}{\beta}(u)$$

and consequently the function is altered at most in its sign. Again replace $2x$ by $4x$ and we have from the last equation

$$\vartheta\binom{\alpha}{\beta}(u+4x)=\vartheta\binom{\alpha}{\beta}(u)$$

or there is no alteration.

Combining all these results we may say that the quadruple theta-functions have the quarter periods (1, 1, 1, 1), the half periods (2, 2, 2, 2) and the whole periods (4, 4, 4, 4).

Write $$(a_{11}a_{12}\ldots a_{44})(x_1x_2x_3x_4)^2=2\Phi$$
and denote the derivative of this with respect to x_i by Φ_{x_i}, we have, of course, $a_{ik}=a_{ki}$ so that the quantities Φ_{x_i} have the values

$$\begin{aligned}\Phi_{x_1}&=a_{11}x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4\\ \Phi_{x_2}&=a_{21}x_1+a_{22}x_2+a_{23}x_3+a_{24}x_4\\ \Phi_{x_3}&=a_{31}x_1+a_{32}x_2+a_{33}x_3+a_{34}x_4\\ \Phi_{x_4}&=a_{41}x_1+a_{42}x_2+a_{43}x_3+a_{44}x_4\end{aligned}$$

Now change u into $u+\frac{1}{\pi i}\cdot\Phi_x$ and examine the function

$$\vartheta\begin{pmatrix}\alpha-x\\ \beta\end{pmatrix}\left(u+\frac{1}{\pi i}\cdot\Phi_x\right)$$

Writing for brevity $$(a_{11}a_{12}\ldots a_{44})=(a)\text{ etc.,}$$
we have for the exponent in the general term the value

$$\tfrac{1}{4}(a)(m+\alpha)^2+\tfrac{1}{2}\pi i\Sigma(m+\alpha)(u+\beta)+A$$

where A is given by the equation

$$\begin{aligned}A=-\tfrac{1}{2}(a)(m+\alpha)(x)\quad&+\tfrac{1}{2}\Sigma(m+\alpha)\Phi_x\\ +\tfrac{1}{4}(a)(x)^2\quad&-\tfrac{1}{2}\pi i\Sigma x(u+\beta)\\ &-\tfrac{1}{2}\Sigma x\Phi_x.\end{aligned}$$

The terms in the right-hand columns are easily seen to be

$$\begin{aligned}&=\tfrac{1}{2}(a)(m+\alpha)(x)\\ &-\tfrac{1}{2}\pi i\Sigma x(u+\beta)\\ &+\tfrac{1}{2}(a)(x)^2\end{aligned}$$

so that we have $$A=-\tfrac{1}{4}(a)(x)^2-\tfrac{1}{2}\pi i\Sigma x(u+\beta)$$
which is independent of m and consequently gives a factor for the entire series. We have then

$$\vartheta\begin{pmatrix}\alpha-x\\ \beta\end{pmatrix}\left(u+\frac{1}{\pi i}\Phi_x\right)=e^A\vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(u)$$

or replacing (since x is an integer) α by $\alpha+x$

$$\vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\left(u+\frac{1}{\pi i}\Phi_x\right)=e^A\vartheta\begin{pmatrix}\alpha+x\\ \beta\end{pmatrix}(u).$$

The change of u into $u+\frac{1}{\pi i}\Phi_x$ then interchanges the functions and affects each of them by the exponential factor e^A. The quantities $\frac{1}{\pi i}\Phi_x$ are called conjoint quarter quasi-periods.

The total number of the p-tuple theta functions is 2^{2p}, so for $p=4$ we have 256 quadruple functions, each of these has a particular for the characteristic

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, = \begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}.$$

The following table gives the entire group of characteristics and suggests a notation for the functions; the upper line of the characteristic is invariable for each column and the lower line for each row.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0000 0000	1000 0000	0100 0000	0010 0000	0001 0000	1100 0000	0110 0000	0011 0000	1001 0000	1010 0000	0101 0000	1110 0000	0111 0000	1011 0000	1101 0000	1111 0000
2	0000 1000	1000 *	0100	0010	0001	1100 *	0110	0011	1001 *	1010 *	0101	1110 *	0111	1011 *	1101 *	1111 *
3	0000 0100	1000	0100 *	0010	0001	1100 *	0110 *	0011	1001	1010	0101 *	1110 *	0111 *	1011	1101 *	1111 *
4	0000 0010	1000	0100	0010 *	0001	1100	0110 *	0011 *	1001	1010 *	0101	1110 *	0111 *	1011 *	1101	1111 *
5	0000 0001	1000	0100	0010	0001 *	1100	0110	0011 *	1001 *	1010	0101 *	1110	0111 *	1011 *	1101 *	1111 *
6	0000 1100	1000 *	0100 *	0010	0001	1100	0110 *	0011	1001 *	1010 *	0101 *	1110	0111 *	1011 *	1101	1111
7	0000 0110	1000	0100 *	0010 *	0001	1100 *	0110	0011 *	1001	1010 *	0101 *	1110	0111	1011 *	1101 *	1111
8	0000 0011	1000	0100	0010 *	0001 *	1100	0110 *	0011	1001 *	1010 *	0101 *	1110 *	0111	1011	1101 *	1111
9	0000 1001	1000 *	0100	0010	0001 *	1100 *	0110	0011 *	1001	1010 *	0101 *	1110 *	0111 *	1011	1101	1111
10	0000 1010	1000 *	0100	0010 *	0001	1100 *	0110 *	0011 *	1001 *	1010	0101	1110	0111 *	1011	1101 *	1111
11	0000 0101	1000	0100 *	0010	0001 *	1100 *	0110 *	0011 *	1001 *	1010	0101	1110 *	0111	1011 *	1101	1111
12	0000 1110	1000 *	0100 *	0010 *	0001	1100	0110	0011 *	1001 *	1010	0101 *	1110 *	0111	1011	1101	1111 *
13	0000 0111	1000	0100 *	0010 *	0001 *	1100 *	0110	0011	1001 *	1010 *	0101	1110	0111 *	1011	1101	1111 *
14	0000 1011	1000 *	0100	0010 *	0001 *	1100 *	0110 *	0011	1001	1010	0101 *	1110	0111	1011 *	1101	1111 *
15	0000 1101	1000 *	0100 *	0010	0001 *	1100	0110 *	0011 *	1001	1010 *	0101	1110	0111	1011	1101 *	1111 *
16	0000 1111	1000 *	0100 *	0010 *	0001 *	1100	0110	0011	1001	1010	0101	1110 *	0111 *	1011 *	1101 *	1111

Where the lower line of the characteristic is left blank, it is understood to be merely a repetition of the lower line in the same row and first column. The asterisks indicate the odd functions, 120 in all, 8 in each row except the first and 8 in each column except the first. Each of the theta-functions may be denoted by ϑ_{ik} where i stands for the number of the row and k for the number of the column. In the following table, the odd functions alone are indicated. It will be readily seen that there is a perfect symmetry in the arrangement

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1																
2		*				*			*	*		*		*	*	*
3			*			*	*				*	*	*		*	*
4				*			*	*		*		*	*	*		*
5					*			*	*		*		*	*	*	*
6		*	*				*		*	*	*		*	*		
7			*	*		*		*		*	*			*	*	
8				*	*		*		*	*	*	*			*	
9		*			*	*		*		*	*	*	*			
10		*		*		*	*	*	*				*		*	
11			*		*	*	*	*	*			*		*		
12		*	*	*				*	*		*	*				*
13			*	*	*	*			*	*			*			*
14		*		*	*	*	*				*			*		*
15		*	*		*		*	*		*					*	*
16		*	*	*	*							*	*	*	*	

for corresponding to any odd function ϑ_{ik}, there is also a function ϑ_{ki}. In the following table I have simply arranged the functions so that there shall be an unbroken diagonal of the odd ones. This arrangement is no longer symmetrical, and it is not possible to make it so, since there are 120 odd functions, and as is easily seen it is only possible to put 15 in a diagonal, there must always be one more on one side of the diagonal than on the other. There are in this table 52 on the left-hand side and 53 on the right, which, together with the 15 in the diagonal, makes up the entire number 120. The numbers in half parenthesis denote the corresponding rows of the two preceding tables.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1																
2		*				*			*	*		*		*	*	*
3			*			*	*				*	*	*		*	*
4				*			*	*		*		*	*	*		*
5					*			*	*		*		*	*	*	*
7) 6			*	*		*		*		*	*			*	*	
6) 7		*	*				*		*	*	*		*	*		
9) 8		*			*	*		*		*	*	*	*			
8) 9				*	*		*		*	*	*	*			*	
13) 10			*	*	*	*			*	*			*			*
12) 11		*	*	*				*	*		*	*				*
11) 12			*		*	*	*	*	*			*		*		
10) 13		*		*		*	*	*	*				*		*	
14		*		*	*	*	*				*			*		*
15		*	*	*	*							*	*	*	*	
16		*	*		*		*	*		*					*	*

Continuing to follow Prof. Cayley, we next find an expression for the product of two of the quadruple functions, the characteristics of which are respectively $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ and $\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}$. As usual the two functions to be multiplied are taken as

$$\vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(u+u') \text{ and } \vartheta\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}(u-u')$$

these standing of course for

$$\vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(u_1+u'_1,\ u_2+u'_2,\ u_3+u'_3,\ u_4+u'_4)$$

and
$$\vartheta\begin{pmatrix}\alpha'_1\,\alpha'_2\,\alpha'_3\,\alpha'_4\\ \beta'_1\,\beta'_2\,\beta'_3\,\beta'_4\end{pmatrix}(u_1-u'_1,\ u_2-u'_2,\ u_3-u'_3,\ u_4-u'_4)$$

By the above symbolical method of writing the argument of the exponential, we have for the first of these

$$\begin{pmatrix}m+\alpha\\ u+u'+\beta\end{pmatrix} \text{ and for the second } \begin{pmatrix}m'+\alpha'\\ u-u'+\beta'\end{pmatrix}.$$

In the product we of course have as exponent the sum of these two quantities, viz., using all the previously indicated abbreviations, this is

$$\tfrac{1}{4}(a)(m + \alpha)^2 + \tfrac{1}{2}\pi i\Sigma(m + \alpha)(u + u' + \beta)$$
$$+ \tfrac{1}{4}(a)(m' + \alpha')^2 + \tfrac{1}{2}\pi i\Sigma(m' + \alpha')(u - u' + \beta').$$

In order to obtain the exact form of the product, Prof. Cayley compares this sum with the sum of the two functions

$$\begin{pmatrix} \mu + \frac{1}{2}(\alpha + \alpha') \\ 2u + \beta + \beta' \end{pmatrix} \text{ and } \begin{pmatrix} \mu' + \frac{1}{2}(\alpha - \alpha') \\ 2u' + \beta - \beta' \end{pmatrix};$$

this is $\quad = \tfrac{1}{4}(2a)(\mu + \tfrac{1}{2}(\alpha + \alpha'))^2 + \tfrac{1}{2}\pi i\Sigma[\mu + \tfrac{1}{2}(\alpha + \alpha')][2u + \beta + \beta']$

$$+ \tfrac{1}{4}(2a)(\mu' + \tfrac{1}{2}(\alpha - \alpha'))^2 + \tfrac{1}{2}\pi i\Sigma[\mu' + \tfrac{1}{2}(\alpha - \alpha')][2u' + \beta - \beta'].$$

The two sums are made identical by writing

$$m_1 + m'_1 = 2\mu_1,\ m_1 - m'_1 = 2\mu'_1,$$
$$m_2 + m'_2 = 2\mu_2,\ m_2 - m'_2 = 2\mu'_2,$$
$$m_3 + m'_3 = 2\mu_3,\ m_3 - m'_3 = 2\mu'_3,$$
$$m_4 + m'_4 = 2\mu_4,\ m_4 - m'_4 = 2\mu'_4.$$

Subject then to the conditions implied in these relations, the product of the two quadruple functions is, when written out in full,

$$= \sum \text{exp.} \begin{pmatrix} \mu_1 + \frac{1}{2}(\alpha_1 + \alpha'_1), & \mu_2 + \frac{1}{2}(\alpha_2 + \alpha'_2), & \mu_3 + \frac{1}{2}(\alpha_3 + \alpha'_3), & \mu_4 + \frac{1}{2}(\alpha_4 + \alpha'_4) \\ 2u_1 + \beta_1 + \beta'_1), & 2u_2 + \beta_2 + \beta'_2, & 2u_3 + \beta_3 + \beta'_3, & 2u_4 + \beta_4 + \beta'_4 \end{pmatrix}$$

$$\times \sum \text{exp.} \begin{pmatrix} \mu'_1 + \frac{1}{2}(\alpha_1 - \alpha'_1), & \mu'_2 + \frac{1}{2}(\alpha_2 - \alpha'_2), & \mu'_3 + \frac{1}{2}(\alpha_3 - \alpha'_3), & \mu'_4 + \frac{1}{2}(\alpha_4 - \alpha'_4) \\ 2u'_1 + \beta_1 - \beta'_1, & 2u'_2 + \beta_2 - \beta'_2, & 2u'_3 + \beta_3 - \beta'_3, & 2u'_4 + \beta_4 - \beta'_4 \end{pmatrix}$$

The even integers m_i and m'_i may be said to be similar when they are both $\equiv 0$ or both $\equiv 2$ (mod. 4), they may be said to be dissimilar when one of them is $\equiv 0$ and the other $\equiv 2$ (mod. 4). There are then in all 16 cases, as shown in the following table. By a pair is meant $(m_i,\ m'_i)$ the same suffix for each letter.

	Number of Cases.
4 pairs similar, ...	1
3 pairs similar, 1 pair dissimilar, ...	4
2 pairs " 2 pairs "	6
1 pair " 3 pairs "	4
4 pairs dissimilar, ...	1
Total number of cases, ...	16

In the first of these cases the μ and μ' are even; in the second case we may write one pair odd and three pairs even (the pairs here referring to the Greek letters), etc. We have then finally for the product

$$\vartheta\binom{\alpha}{\beta}(u+u').\vartheta\binom{\alpha'}{\beta'}(u-u')=$$

$$\sum\Theta\binom{\frac{1}{2}(\alpha+\alpha')+p}{\beta+\beta'}(2u).\Theta\binom{\frac{1}{2}(\alpha-\alpha')+p}{\beta-\beta'}(2u');$$

here Θ is written instead of ϑ to denote that each a_{ik} is replaced by $2a_{ik}$ and the summation refers to the different values of p, given in the following table, viz.

p_1	p_2	p_3	p_4
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	1	0	0
0	1	1	0
0	0	1	1
1	0	0	1
1	0	1	0
0	1	0	1
1	1	1	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	1

There are in all 136 even quadruple functions, these do not vanish when the arguments u_1, u_2, u_3 and u_4 are made $=0$, and consequently there are 136 constants corresponding to the zero values of the arguments. These constants may be denoted by c_{ik} and are given in the following table. Of course each c here written is understood to be c_{ik} where i and k denote respectively the number of the row and the number of the column. The asterisks are written instead of zeros to denote the zero values of the odd functions. It will be noticed here that both the asterisks and the c's are symmetrically arranged in the table. When u_1, u_2, u_3 and u_4 are each indefinitely small, each of the even functions is reduced to its zero value plus a quadric function of (u) and each of the odd functions are given by linear functions of (u). In the first of these cases we may write

$$\vartheta=c+(c'_{11},\ c'_{12},\ c'_{13},\ c'_{14},\ c'_{22},\ c'_{23},\ c'_{24},\ c'_{33},\ c'_{34},\ c'_{44}\between u_1,\ u_2,\ u_3,\ u_4)^2$$

giving thus 1360 new constants c'_{ik}; in the second case we may write

$$\vartheta = (c''_1,\ c''_2,\ c''_3,\ c''_4 \between u_1,\ u_2,\ u_3,\ u_4)$$

giving 480 new constants c''_i.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
2	c	*	c	c	c	*	c	c	*	*	c	*	c	*	*	
3	c	c	*	c	c	*	*	c	c	c	*	*	*	c	*	*
4	c	c	c	*	c	c	*	*	c	*	c	*	*	*	c	*
5	c	c	c	c	*	c	c	*	*	c	c	*	*	*	*	*
6	c	*	*	c	c	c	*	c	*	*	*	c	*	*	c	c
7	c	c	*	*	c	*	c	*	c	*	*	c	c	*	*	c
8	c	c	c	*	*	c	*	c	*	*	*	*	c	c	*	c
9	c	*	c	c	*	*	c	*	c	*	*	*	*	c	c	c
10	c	*	c	*	c	*	*	*	*	c	c	c	*	c	*	c
11	c	c	*	c	*	*	*	*	*	c	c	*	c	*	c	c
12	c	*	*	*	c	c	c	*	*	c	*	*	c	c	c	*
13	c	c	*	*	*	*	c	c	*	*	c	c	*	c	c	*
14	c	*	c	*	*	*	*	c	c	c	*	c	c	*	c	*
15	c	*	*	c	*	c	*	*	c	*	c	c	c	c	*	*
16	c	*	*	*	*	c	c	c	c	c	c	*	*	*	*	c

A complete product table for the quadruple theta-functions would be much too long to give, as it would contain 256 × 256 or 2^{16} products These would consist of what might be called a square-set of 256 products and 255 other sets each containing 256 products. By a square-set is meant a set consisting of products of the form $\vartheta_{ik}(u + u').\vartheta_{ik}(u - u')$; the remaining sets would have different suffixes for each ϑ. In the following table I give the products contained in the square-set, and instead of writing Θ with its proper suffix I write (following Prof. Cayley) X with a certain suffix; I give first the definition of the X's which refer to $2u_1$, $2u_2$, $2u_3$, $2u_4$; the accented X's refer of course to the Θ-functions with arguments $2u'_1$, $2u'_2$, $2u'_3$, $2u'_4$. In the remaining 255 sets it would be necessary to introduce some new symbols, inasmuch as the characteristics would not all be made up of 0 and 1.

Θ	0000	1000	0100	0010	0001	1100	0110	0011	1001	1010	0101	1110	0111	1011	1101	1111
0000																
1000																
0100																
0010																
0001																
1100																
0110																
0011																
1001																
1010																
0101																
1110																
0111																
1011																
1101																
1111																

=	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}
1																
2																
3																
4																
5																
6																
7																
8																
9																
10																
11																
12																
13																
14																
15																
16																

From this we have, for example

$$\vartheta\begin{pmatrix}0100\\1001\end{pmatrix}=X_{3,9},\ \vartheta\begin{pmatrix}1110\\0110\end{pmatrix}=X_{12,7},\ \vartheta\begin{pmatrix}1111\\1111\end{pmatrix}=X_{16,16},\ \text{etc.}$$

Instead of writing X_{ik} it will occasionally be a little more convenient to write X^i_k, the upper letter denoting the number of the row and the lower one the number of the column. In the following table giving the square-set products, I have written in the first column the suffixes belonging to the particular ϑ; at the top of the next 16 columns will be found the letter x; this is to be read as XX'; the last column contains the first suffix for both X and X' and the first *row* contains the second suffix belonging to X and X', viz. the figure on the left of the comma is the second suffix for X and the figure on the right of the comma is the second suffix for X'. The signs alone appear in every row after the first, the suffixes in the first row being the same, of course, for all of the sixteen equations in each sub-set. The table is divided into 16 different sub-sets; in the first sub-set the general term is $X_{ik}X'_{ik}$, *i. e.* the suffixes are the same for X and X'. This is not the case with the other fifteen sub-sets, but it will be noticed that if i be the number of the sub-set the first row and i^{th} column is always made up of *plus* signs. I give here a few examples illustrating the use of the tables.

$$\begin{aligned}\vartheta_{1,1}\vartheta_{1,1},=\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}\vartheta\begin{pmatrix}0000\\0000\end{pmatrix},=\ &X_{1,1}X'_{1,1}+X_{1,2}X'_{1,2}+X_{1,3}X'_{1,3}+X_{1,4}X'_{1,4}\\&+X_{1,5}X'_{1,5}+X_{1,6}X'_{1,6}+X_{1,7}X'_{1,7}+X_{1,8}X'_{1,8}\\&+X_{1,9}X'_{1,9}+X_{1,10}X'_{1,10}+X_{1,11}X'_{1,11}+X_{1,12}X'_{1,12}\\&+X_{1,13}X'_{1,13}+X_{1,14}X'_{1,14}+X_{1,15}X'_{1,15}+X_{1,16}X'_{1,16},\end{aligned}$$

$$\begin{aligned}\vartheta_{4,5}\vartheta_{4,5},=\vartheta\begin{pmatrix}0001\\0010\end{pmatrix}\vartheta\begin{pmatrix}0001\\0010\end{pmatrix},=\ &X_{1,5}X'_{1,1}+X_{1,9}X'_{1,2}+X_{1,11}X'_{1,3}-X_{1,8}X'_{1,4}\\&+X_{1,1}X'_{1,5}+X_{1,15}X'_{1,6}-X_{1,13}X'_{1,7}-X_{1,4}X'_{1,8}\\&+X_{1,2}X'_{1,9}-X_{1,14}X'_{1,10}+X_{1,3}X'_{1,11}-X_{1,16}X'_{1,12}\\&-X_{1,7}X'_{1,13}-X_{1,10}X'_{1,14}+X_{1,6}X'_{1,15}-X_{1,12}X'_{1,16},\end{aligned}$$

$$\begin{aligned}\vartheta_{10,12}\vartheta_{10,12},=\vartheta\begin{pmatrix}1110\\1010\end{pmatrix}\vartheta\begin{pmatrix}1110\\1010\end{pmatrix},=\ &X_{1,12}X'_{1,1}-X_{1,7}X'_{1,2}+X_{1,10}X'_{1,3}-X_{1,6}X'_{1,4}\\&+X_{1,16}X'_{1,5}-X_{1,4}X'_{1,6}-X_{1,2}X'_{1,7}-X_{1,15}X'_{1,8}\\&-X_{1,13}X'_{1,9}+X_{1,3}X'_{1,10}+X_{1,14}X'_{1,11}+X_{1,1}X'_{1,12}\\&-X_{1,9}X'_{1,13}+X_{1,11}X'_{1,14}-X_{1,8}X'_{1,15}+X_{1,5}X'_{1,16}.\end{aligned}$$

I

ϑ	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x	Suff.
1,1	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	1
2,1	+1	−2	+3	+4	+5	−6	+7	+8	−9	−10	+11	−12	+13	−14	−15	−16	1
3,1	+1	+2	−3	+4	+5	−6	−7	+8	+9	+10	−11	−12	−13	+14	−15	−16	1
4,1	+1	+2	+3	−4	+5	+6	−7	−8	+9	−10	+11	−12	−13	−14	+15	−16	1
5,1	+1	+2	+3	+4	−5	+6	+7	−8	−9	+10	−11	+12	−13	−14	−15	−16	1
6,1	+1	−2	−3	+4	+5	+6	−7	+8	−9	−10	−11	+12	−13	−14	+15	+16	1
7,1	+1	+2	−3	−4	+5	−6	+7	−8	+9	−10	−11	+12	+13	−14	−15	+16	1
8,1	+1	+2	+3	−4	−5	+6	−7	+8	−9	−10	−11	−12	+13	+14	−15	+16	1
9,1	+1	−2	+3	+4	−5	−6	+7	−8	+9	−10	−11	−12	−13	+14	+15	+16	1
10,1	+1	−2	+3	−4	+5	−6	−7	−8	−9	+10	+11	+12	−13	+14	−15	+16	1
11,1	+1	+2	−3	+4	−5	−6	−7	−8	−9	+10	+11	−12	+13	−14	+15	+16	1
12,1	+1	−2	−3	−4	+5	+6	+7	−8	−9	+10	−11	−12	+13	+14	+15	−16	1
13,1	+1	+2	−3	−4	−5	−6	+7	+8	−9	−10	+11	+12	−13	+14	+15	−16	1
14,1	+1	−2	+3	−4	−5	−6	−7	+8	+9	+10	−11	+12	+13	−14	+15	−16	1
15,1	+1	−2	−3	+4	−5	+6	−7	−8	+9	−10	+11	+12	+13	+14	−15	−16	1
16,1	+1	−2	−3	−4	−5	+6	+7	+8	+9	+10	+11	−12	−13	−14	−15	+16	1

II

	ϑ		x	x	x	x	x	x	x	x	x	x	x.	x	x	x	x	Suff.
	1,2	+ 2,1	+ 1,2	+ 6,3	+ 10,4	+ 9,5	+ 3,6	+ 12,7	+ 14,8	+ 5,9	+ 4,10	+ 15,11	+ 7,12	+ 16,13	+ 8,14	+ 11,15	+ 13,16	1
*	2,2	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
	3,2	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
	4,2	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
	5,2	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6,2	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,2	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
	8,2	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,2	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,2	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,2	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
*	12,2	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,2	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
*	14,2	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
*	15,2	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,2	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

III

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,3	+ 3,1	+ 6,2	+ 1,3	+ 7,4	+ 11,5	+ 2,6	+ 4,7	+ 13,8	+ 15,9	+ 12,10	+ 5,11	+ 10,12	+ 8,13	+ 16,14	+ 9,15	+ 14,16	1
	2,3	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3,3	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
	4,3	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
	5,3	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6,3	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
*	7,3	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
	8,3	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
	9,3	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
	10,3	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,3	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,3	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
*	13,3	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
	14,3	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,3	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,3	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

IV

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,4	+ 4,1	+ 10,2	+ 7,3	+ 1,4	+ 8,5	+ 12,6	+ 3,7	+ 5,8	+ 14,9	+ 2,10	+ 13,11	+ 6,12	+ 11,13	+ 9,14	+ 16,15	+ 15,16	1
	2,4	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
	3,4	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
*	4,4	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5,4	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
	6,4	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,4	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
*	8,4	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,4	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,4	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,4	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
*	12,4	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
*	13,4	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
*	14,4	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
	15,4	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16,4	−	+	+	+	+	−	−	−	−	−	-	+	+	+	+	−	1

V

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,5	+ 5,1	+ 9,2	+ 11,3	+ 8,4	+ 1,5	+ 15,6	+ 13,7	+ 4,8	+ 2,9	+ 14,10	+ 3,11	+ 16,12	+ 7,13	+ 10,14	+ 6,15	+ 12,16	1
	2,5	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
	3,5	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
	4,5	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
*	5,5	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
	6,5	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
	7,5	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,5	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
*	9,5	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
	10,5	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,5	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
	12,5	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
*	13,5	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
*	14,5	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
*	15,5	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,5	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

VI

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,6	+ 6,1	+ 3,2	+ 2,3	+ 12,4	+ 15,5	+ 1,6	+ 10,7	+ 16,8	+ 11,9	+ 7,10	+ 9,11	+ 4,12	+ 14,13	+ 13,14	+ 5,15	+ 8,16	1
* 2,6	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
* 3,6	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
4,6	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
5,6	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
6,6	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
* 7,6	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
8,6	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
* 9,6	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
* 10,6	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
* 11,6	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
12,6	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
* 13,6	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
* 14,6	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
15,6	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
16.6	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

VII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,7	+ 7,1	+ 12,2	+ 4,3	+ 3,4	+ 13,5	+ 10,6	+ 1,7	+ 11,8	+ 16,9	+ 6,10	+ 8,11	+ 2,12	+ 5,13	+ 15,14	+ 14,15	+ 9,16	1
	2,7	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3,7	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
*	4,7	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5,7	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6,7	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,7	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,7	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,7	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,7	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11,7	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
	12,7	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
	13,7	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
*	14,7	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
*	15,7	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
	16,7	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

VIII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,8	+ 8,1	+ 14,2	+ 13,3	+ 5,4	+ 4,5	+ 16,6	+ 11,7	+ 1,8	+ 10,9	+ 9,10	+ 7,11	+ 15,12	+ 3,13	+ 2,14	+ 12,15	+ 6,16	1
	2,8	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
	3,8	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
*	4,8	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
*	5,8	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
	6,8	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,8	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
	8,8	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,8	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,8	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11,8	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,8	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,8	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
	14,8	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,8	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
	16,8	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

IX

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,9	+ 9,1	+ 5,2	+ 15,3	+ 14,4	+ 2,5	+ 11,6	+ 16,7	+ 10,8	+ 1,9	+ 8,10	+ 6,11	+ 13,12	+ 12,13	+ 4,14	+ 3,15	+ 7,16	1
*	2,9	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
	3,9	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
	4,9	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
*	5,9	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
*	6,9	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,9	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,9	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,9	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,9	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11,9	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,9	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
*	13,9	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
	14,9	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
	15,9	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
	16,9	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

X

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,10	+ 10,1	+ 4,2	+ 12,3	+ 2,4	+ 14,5	+ 7,6	+ 6,7	+ 9,8	+ 8,9	+ 1,10	+ 16,11	+ 3,12	+ 15,13	+ 5,14	+ 13,15	+ 11,16	1
* 2,10	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
3,10	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
* 4,10	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
5,10	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
* 6,10	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
* 7,10	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
* 8,10	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
* 9,10	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
10,10	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
11,10	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
12,10	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
* 13,10	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
14,10	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
* 15,10	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
16.10	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

XI

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,11	+ 11,1	+ 15,2	+ 5,3	+ 13,4	+ 3,5	+ 9,6	+ 8,7	+ 7,8	+ 6,9	+ 16,10	+ 1,11	+ 14,12	+ 4,13	+ 12,14	+ 2,15	+ 10,16	1
	2,11	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3,11	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
	4,11	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
	5,11	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
*	6,11	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
*	7,11	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
*	8,11	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
*	9,11	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
	10,11	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
	11,11	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
*	12,11	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,11	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
*	14,11	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
	15,11	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
	16,11	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

XII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,12	+ 12,1	+ 7,2	+ 10,3	+ 6,4	+ 16,5	+ 4,6	+ 2,7	+ 15,8	+ 13,9	+ 3,10	+ 14,11	+ 1,12	+ 9,13	+ 11,14	+ 8,15	+ 5,16	1
*	2,12	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
*	3,12	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
*	4,12	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5,12	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
	6,12	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
	7,12	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,12	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
*	9,12	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
	10,12	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,12	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,12	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,12	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
	14,12	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
	15,12	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16,12	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XIII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,13	+ 13,1	+ 16,2	+ 8,3	+ 11,4	+ 7,5	+ 14,6	+ 5,7	+ 3,8	+ 12,9	+ 15,10	+ 4,11	+ 9,12	+ 1,13	+ 6,14	+ 10,15	+ 2,16	1
	2,13	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3,13	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
*	4,13	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
*	5,13	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
*	6,13	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,13	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
	8,13	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,13	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,13	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,13	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
	12,13	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
*	13,13	−	−	+	+	+	−	+	−	+	+	−	−	+	−	−	+	1
	14,13	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
	15,13	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16,13	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XIV

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,14	+ 14,1	+ 8,2	+ 16,3	+ 9,4	+ 10,5	+ 13,6	+ 15,7	+ 2,8	+ 4,9	+ 5,10	+ 12,11	+ 11,12	+ 6,13	+ 1,14	+ 7,15	+ 3,16	1
* 2,14	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
3,14	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
* 4,14	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
* 5,14	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
* 6,14	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
* 7,14	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
8,14	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
9,14	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
10,14	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
* 11,14	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
12,14	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
13,14	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
* 14,14	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
15,14	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
* 16.14	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XV

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,15	+ 15,1	+ 11,2	+ 9,3	+ 16,4	+ 6,5	+ 5,6	+ 14,7	+ 12,8	+ 3,9	+ 13,10	+ 2,11	+ 8,12	+ 10,13	+ 7,14	+ 1,15	+ 4,16	1
*	2,15	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
*	3,15	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
	4,15	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
*	5,15	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
	6,15	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,15	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
*	8,15	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,15	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,15	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,15	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
	12,15	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
	13,15	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
	14,15	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,15	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,15	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XVI

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,16	+ 16,1	+ 13,2	+ 14,3	+ 15,4	+ 12,5	+ 8,6	+ 9,7	+ 6,8	+ 7,9	+ 11,10	+ 10,11	+ 5,12	+ 2,13	+ 3,14	+ 4,15	+ 1,16	1
* 2,16	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
* 3,16	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
* 4,16	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
* 5,16	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
6,16	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
7,16	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
8,16	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
9,16	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
10,16	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
11,16	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
* 12,16	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
* 13,16	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
* 14,16	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
* 15,16	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
16.16	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

There is a certain symmetry in these tables, although it is hardly apparent in this (the most convenient) way of writing them. In the first table, the signs are all the same in the first row and in the first column; leaving out these lines and taking the fifteen by fifteen square which is left, it will be noticed that the signs are arranged in the same way in both the first row and the first column; forming again the fourteen by fourteen square, etc., the same thing will be noticed to hold true. I do not see any way of arranging the next fifteen tables in order to bring them into this form.

By making the quantities u'_1, u'_2, u'_3 and u'_4 all equal to zero, we have a linear function of the X_1, X_2, X_3, etc., equal to the square of a certain ϑ; and further making the u_1, u_2, u_3 and u_4 vanish, we have the constants c_{ik} expressed as quadratic functions of the zero values of X_{ik} and X'_{ik} (the zero value of X_{ik} being of course equal to the zero value of X'_{ik}). The more general form for the products is indicated by Nöther in his memoir above referred to, where he shows that the product

$$\vartheta(u+v+w).\vartheta(u-v)$$

can be expressed linearly and homogeneously by sixteen products of the form

$$\vartheta_a(u+w).\vartheta_a(u).$$

This he obtains from a general theorem in the theory of the ϑ-functions of any number of arguments, viz. (for four arguments) "*that between any* $2^4+1=17$ *products of the form*

$$\vartheta_a(u+v')\vartheta_a(u+v'')$$

which have equal arguments u_1, u_2, u_3, u_4, *and equal sums*

$$v'_1+v''_1,\quad v'_2+v''_2,\quad v'_3+v''_3,\quad v'_4+v''_4,$$

there exists a linear homogeneous relation, the coefficients of which are independent of u." The sums are in the present case all equal zero, as $v''=-v'$, $=u'$.

The zero values of X_i and X'_i being the same, they may be represented by α_i. In the following tables are arranged the squares of the zero values of the theta functions which are denoted by c_{ik}, only the even functions are written down, since the odd functions are zero for zero values of the arguments. The first table containing 16 rows corresponds to table I above; the second, third, etc., tables containing only 8 rows each, correspond to tables II, III, etc., above. The first column contains the suffixes of each c and the following columns contain the suffixes of the α's. The sign of each term is written above it just as in the preceding tables, of course the arrangement of signs is the same as in these tables. When two suffixes are separated by a comma, the product of the corresponding α's is meant; thus $\boxed{\overset{+}{i,k}}$ is to be read $+\alpha_i . \alpha_k$. Where there is only one suffix written down, the square of the corresponding α is meant; thus $\boxed{\overset{+}{i}}$ is to be read $+\alpha_i^2$. These remarks do not of course apply to the first column, as this contains the regular suffixes of each ϑ as before defined, the comma is there written simply for convenience in reading. Another set of 16 tables might be given here giving the squared values of each ϑ_{ik} in terms of X_i and α_i, viz. as already mentioned linear functions of the X_i. This is not necessary, however, as these relations can be obtained at once from the last 16 tables by simply reading for x, αX instead of XX'. Another point is to be noted in using these tables, viz. that $\boxed{\overset{\pm}{i,k}}$ has the same value as $\boxed{\overset{\pm}{k,i}}$. It will be observed then in every table after the first that there are only eight terms, each product $\alpha_i . \alpha_k$ coming in twice and each product $\alpha_i . \alpha_k$ having the same sign as $\alpha_k . \alpha_i$. The odd functions in the previous set of tables are marked by an asterisk at the left-hand side of the characteristics contained in the first column; for these functions it will be observed that to each $\pm\alpha_i . \alpha_k$ there corresponds $\mp\alpha_i . \alpha_k$ in another column—the odd values, therefore, vanishing for zero values of the arguments, or in the general case (referring of course only to the odd functions) there is corresponding to any term of the form $\pm X_i , X_k'$ another term of the form $\mp X_k , X_i'$. Table I contains only sums and differences of *squares* of α_i, and the other tables contain only products of the form $2\alpha_i . \alpha_k$. This arrangement differs slightly from that given by Prof. Cayley for the double theta-functions. The construction of tables giving the values of $c_i . c_k$ would obviously involve a great deal of work, and from the difficulty in using them would be practically of no value.

1

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,1	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16
2,1	+1	−2	+3	+4	+5	−6	+7	+8	−9	−10	+11	−12	+13	−14	−15	−16
3,1	+1	+2	−3	+4	+5	−6	−7	+8	+9	+10	−11	−12	−13	+14	−15	−16
4,1	+1	+2	+3	−4	+5	+6	−7	−8	+9	−10	+11	−12	−13	−14	+15	−16
5,1	+1	+2	+3	+4	−5	+6	+7	−8	−9	+10	−11	+12	−13	−14	−15	−16
6,1	+1	−2	−3	+4	+5	+6	−7	+8	−9	−10	−11	+12	−13	−14	+15	+16
7,1	+1	+2	−3	−4	+5	−6	+7	−8	+9	−10	−11	+12	+13	−14	−15	+16
8,1	+1	+2	+3	−4	−5	+6	−7	+8	−9	−10	−11	−12	+13	+14	−15	+16
9.1	+1	−2	+3	+4	−5	−6	+7	−8	+9	−10	−11	−12	−13	+14	+15	+16
10,1	+1	−2	+3	−4	+5	−6	−7	−8	−9	+10	+11	+12	−13	+14	−15	+16
11,1	+1	+2	−3	+4	−5	−6	−7	−8	−9	+10	+11	−12	+13	−14	+15	+16
12,1	+1	−2	−3	−4	+5	+6	+7	−8	−9	+10	−11	−12	+13	+14	+15	−16
13,1	+1	+2	−3	−4	−5	−6	+7	+8	−9	−10	+11	+12	−13	+14	+15	−16
14,1	+1	−2	+3	−4	−5	−6	−7	+8	+9	+10	−11	+12	+13	−14	+15	−16
15,1	+1	−2	−3	+4	−5	+6	−7	−8	+9	−10	+11	+12	+13	+14	−15	−16
16,1	+1	−2	−3	−4	−5	+6	+7	+8	+9	+10	+11	−12	−13	−14	−15	+16

2

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,2	+ 2,1	+ 1,2	+ 6,3	+ 10,4	+ 9,5	+ 3,6	+ 12,7	+ 14,8	+ 5,9	+ 4,10	+ 15,11	+ 7,12	+ 16,13	+ 8,14	+ 11,15	+ 13,16
3,2	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
4,2	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
5,2	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
7,2	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
8,2	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
11,2	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
13,2	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−

3

	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,3	+ 3,1	+ 6,2	+ 1,3	+ 7,4	+ 11,5	+ 2,6	+ 4,7	+ 13,8	+ 15,9	+ 12,10	+ 5,11	+ 10,12	+ 8.13	+ 16,14	+ 9,15	+ 14,16
2,3	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
4,3	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
5,3	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
8,3	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
9,3	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
10,3	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
14,3	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−

4

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,4	+ 4,1	+ 10,2	+ 7,3	+ 1,4	+ 8,5	+ 12,6	+ 3,7	+ 5,8	+ 14,9	+ 2,10	+ 13,11	+ 6,12	+ 11,13	+ 9,14	+ 16,15	+ 15,16
2,4	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
3,4	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
5,4	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
6,4	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
9,4	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
11,4	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
15,4	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

5

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,5	+ 5,1	+ 9,2	+ 11,3	+ 8,4	+ 1,5	+ 15,6	+ 13,7	+ 4,8	+ 2,9	+ 14,10	+ 3,11	+ 16,12	+ 7,13	+ 10,14	+ 6,15	+ 12,16
2,5	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
3,5	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
4,5	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
6,5	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
7,5	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
10,5	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
12,5	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−

6

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,6	+ 6,1	+ 3,2	+ 2,3	+ 12,4	+ 15,5	+ 1,6	+ 10,7	+ 16,8	+ 11,9	+ 7,10	+ 9,11	+ 4,12	+ 14,13	+ 13,14	+ 5,15	+ 8,16
4,6	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
5,6	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
6,6	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
8,6	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
12,6	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
15,6	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−
16,6	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

7

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,7	+ 7,1	+ 12,2	+ 4,3	+ 3,4	+ 13,5	+ 10,6	+ 1,7	+ 11,8	+ 16,9	+ 6,10	+ 8,11	+ 2,12	+ 5,13	+ 15,14	+ 14,15	+ 9,16
2,7	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
5,7	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
7,7	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
9,7	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
12,7	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
13,7	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
16,7	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

8

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,8	+ 8,1	+ 14,2	+ 13,3	+ 5,4	+ 4,5	+ 16,6	+ 11,7	+ 1,8	+ 10,9	+ 9,10	+ 7,11	+ 15,12	+ 3,13	+ 2,14	+ 12,15	+ 6,16
2,8	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
3,8	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
6,8	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
8,8	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
13,8	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
14,8	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
16,8	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

9

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,9	+ 9,1	+ 5,2	+ 15,3	+ 14,4	+ 2,5	+ 11,6	+ 16,7	+ 10,8	+ 1,9	+ 8,10	+ 6,11	+ 13,12	+ 12,13	+ 4,14	+ 3,15	+ 7,16
3,9	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
4,9	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
7,9	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
9,9	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
14,9	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
15,9	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−
16,9	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

10

c^2	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
1,10	+ 10,1	+ 4,2	+ 12,3	+ 2,4	+ 14,5	+ 7,6	+ 6,7	+ 9,8	+ 8,9	+ 1,10	+ 16,11	+ 3,12	+ 15,13	+ 5,14	+ 13,15	+ 11,16
3,10	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
5,10	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
10,10	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
11,10	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
12,10	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
14,10	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
16,10	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

11

c^2	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
1,11	+ 11,1	+ 15,2	+ 5,3	+ 13,4	+ 3,5	+ 9,6	+ 8,7	+ 7,8	+ 6,9	+ 16,10	+ 1,11	+ 14,12	+ 4,13	+ 12,14	+ 2,15	+ 10,16
2,11	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
4,11	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
10,11	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
11,11	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
13,11	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
15,11	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−
16,11	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

12

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,12	+ 12,1	+ 7,2	+ 10,3	+ 6,4	+ 16,5	+ 4,6	+ 2,7	+ 15,8	+ 13,9	+ 3,10	+ 14,11	+ 1,12	+ 9,13	+ 11,14	+ 8,15	+ 5,16
5,12	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
6,12	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
7,12	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
10,12	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
13,12	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
14,12	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
15,12	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

13

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,13	+ 13,1	+ 16,2	+ 8,3	+ 11,4	+ 7,5	+ 14,6	+ 5,7	+ 3,8	+ 12,9	+ 15,10	+ 4,11	+ 9,12	+ 1,13	+ 6,14	+ 10,15	+ 2,16
2,13	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
7,13	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
8,13	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
11,13	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
12,13	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
14,13	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
15,13	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

14

c^2	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
1,14	+ 14,1	+ 8,2	+ 16,3	+ 9,4	+ 10,5	+ 13,6	+ 15,7	+ 2,8	+ 4,9	+ 5,10	+ 12,11	+ 11,12	+ 6,13	+ 1,14	+ 7,15	+ 3,16
3,14	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
8,14	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
9,14	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
10,14	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
12,14	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
13,14	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
15,14	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

15

c^2	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
1,15	+ 15,1	+ 11,2	+ 9,3	+ 16,4	+ 6,5	+ 5,6	+ 14,7	+ 12,8	+ 3,9	+ 13,10	+ 2,11	+ 8,12	+ 10,13	+ 7,14	+ 1,15	+ 4,16
4,15	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
6,15	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
9,15	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
11,15	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
12,15	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
13,15	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
14,15	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−

16

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,16	+ 16,1	+ 13,2	+ 14,3	+ 15,4	+ 12,5	+ 8,6	+ 9,7	+ 6,8	+ 7,9	+ 11,10	+ 10,11	+ 5,12	+ 2,13	+ 3,14	+ 4,15	+ 1,16
6,16	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
7,16	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
8,16	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
9,16	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
10,16	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
11,16	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
16,16	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

The following examples will serve to illustrate the method of using the tables.

TABLE 1.

$$c^2_{1,1} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \alpha_5^2 + \alpha_6^2 + \alpha_7^2 + \alpha_8^2 + \alpha_9^2 + \alpha_{10}^2 + \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \alpha_{14}^2 + \alpha_{15}^2 + \alpha_{16}^2$$

$$c^2_{4,1} = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \alpha_4^2 + \alpha_5^2 + \alpha_6^2 - \alpha_7^2 - \alpha_8^2 + \alpha_9^2 - \alpha_{10}^2 + \alpha_{11}^2 - \alpha_{12}^2 - \alpha_{13}^2 - \alpha_{14}^2 + \alpha_{15}^2 - \alpha_{16}^2$$

$$c^2_{9,1} = \alpha_1^2 - \alpha_2^2 + \alpha_3^2 + \alpha_4^2 - \alpha_5^2 - \alpha_6^2 + \alpha_7^2 - \alpha_8^2 + \alpha_9^2 - \alpha_{10}^2 - \alpha_{11}^2 - \alpha_{12}^2 - \alpha_{13}^2 + \alpha_{14}^2 + \alpha_{15}^2 + \alpha_{16}^2$$

$$c^2_{16,1} = \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - \alpha_4^2 - \alpha_5^2 + \alpha_6^2 + \alpha_7^2 + \alpha_8^2 + \alpha_9^2 + \alpha_{10}^2 + \alpha_{11}^2 - \alpha_{12}^2 - \alpha_{13}^2 - \alpha_{14}^2 - \alpha_{15}^2 + \alpha_{16}^2 .$$

TABLE 4.

$$c^2_{1,4} = 2(\alpha_4\alpha_1 + \alpha_{10}\alpha_2 + \alpha_7\alpha_3 + \alpha_1\alpha_4 + \alpha_8\alpha_5 + \alpha_{12}\alpha_6 + \alpha_{14}\alpha_9 + \alpha_{13}\alpha_{11})$$

$$c^2_{11,4} = 2(\alpha_4\alpha_1 + \alpha_{10}\alpha_2 - \alpha_7\alpha_3 + \alpha_1\alpha_4 - \alpha_8\alpha_5 - \alpha_{12}\alpha_6 - \alpha_{14}\alpha_9 + \alpha_{13}\alpha_{11}) .$$

TABLE 10.

$$c^2_{5,10} = 2(\alpha_{10}\alpha_1 + \alpha_4\alpha_2 + \alpha_{12}\alpha_3 - \alpha_{14}\alpha_5 + \alpha_7\alpha_6 - \alpha_9\alpha_8 - \alpha_{16}\alpha_{11} - \alpha_{15}\alpha_{13})$$

$$c^2_{16,10} = 2(\alpha_{10}\alpha_1 - \alpha_4\alpha_2 - \alpha_{12}\alpha_3 - \alpha_{14}\alpha_5 + \alpha_7\alpha_6 + \alpha_9\alpha_8 + \alpha_{16}\alpha_{11} - \alpha_{15}\alpha_{13}) .$$

TABLE 15.

$$c^2_{6,15} = 2(\alpha_{15}\alpha_1 - \alpha_{11}\alpha_2 - \alpha_9\alpha_3 + \alpha_{16}\alpha_4 + \alpha_6\alpha_5 - \alpha_{14}\alpha_7 + \alpha_{12}\alpha_8 - \alpha_{13}\alpha_{10})$$

$$c^2_{12,15} = 2(\alpha_{15}\alpha_1 - \alpha_{11}\alpha_2 - \alpha_9\alpha_3 - \alpha_{16}\alpha_4 + \alpha_6\alpha_5 + \alpha_{14}\alpha_7 - \alpha_{12}\alpha_8 + \alpha_{13}\alpha_{10}) .$$

The relations connecting these zero values would be determined with considerable difficulty from the above tables—the difficulty lying in the fact of the great number of terms in the α's which go to make up a given c. The following relation is obtained, however, without much trouble, viz.

$$\begin{vmatrix} c^2_{1,1}, & c^2_{2,1}, & c^2_{3,1}, & c^2_{4,1} \\ c^2_{5,1}, & c^2_{6,1}, & c^2_{7,1}, & c^2_{8,1} \\ c^2_{9,1}, & c^2_{10,1}, & c^2_{11,1}, & c^2_{12,1} \\ c^2_{13,1}, & c^2_{14,1}, & c^2_{15,1}, & c^2_{16,1} \end{vmatrix} = 16 \begin{vmatrix} \alpha^2_{16}, & \alpha^2_{15}, & \alpha^2_{14}, & \alpha^2_{13} \\ \alpha^2_{12}, & \alpha^2_{11}, & \alpha^2_{10}, & \alpha^2_{9} \\ \alpha^2_{8}, & \alpha^2_{7}, & \alpha^2_{6}, & \alpha^2_{5} \\ \alpha^2_{4}, & \alpha^2_{3}, & \alpha^2_{2}, & \alpha^2_{1} \end{vmatrix}$$

corresponding to the relation, in Prof. Cayley's notation for the double theta-functions

$$\begin{vmatrix} c^2_0, & c^2_4 \\ c^2_8, & c^2_{12} \end{vmatrix} = \begin{vmatrix} \delta^2, & \gamma^2 \\ \beta^2, & \alpha^2 \end{vmatrix}, \; c^2_{15} c^2_3.$$

The suffixes in this notation should each be increased by unity to make them correspond to the notation which I have used, and the $\alpha, \beta, \gamma, \delta$ should be replaced by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

Brioschi, in his paper above referred to, has given some general formulas connected with the extension of the Göpel biquadratic relations between double theta-functions to the case of functions of n arguments. These relations, for the present case of four arguments, could, of course, be directly obtained from the tables given above, together with the remaining set of product tables, which would obviously require too much space to write down. I only mention this paper of Brioschi's which, though brief and in some respects unsatisfactory, is the only one I know of in which an attempt is made to generalize the Göpel relations to the case of any number of arguments. Another brief paper of Brioschi's, which I have just seen, is contained in the Atti della R. Acad. dei Lincei, serie terza, Vol. vii, the title is "Le relazioni algebriche fra le funzioni iperellittiche del primo ordine," which, though referring directly only to the double theta-functions, contains some valuable hints as to methods of procedure in the case of functions of more than two arguments. Prof. Cayley gives, in his memoir (page 941), a well known relation between the zero values of the even double theta-functions, viz.

$$\begin{vmatrix} c^2_{12}, & c^2_1, & c^2_6 \\ c^2_9, & -c^2_4, & c^2_3 \\ c^2_2, & -c^2_{15}, & -c^2_8 \end{vmatrix} \div c^2_0 = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

or in my notation

$$\begin{vmatrix} c^2_{13}, & c^2_2, & c^2_7 \\ c^2_{10}, & -c^2_5, & c^2_4 \\ c^2_3, & -c^2_{16}, & -c^2_9 \end{vmatrix} \div c^2_1 = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

where the a, b, c, a', . . . , c'' constitute a system of coefficients in the transformation between two sets of rectangular coordinates. I have tried to find something similar for the case of the quadruple functions, but so far have not succeeded. It is evident that a relation identical in form with the above cannot exist, inasmuch as there are 9, *i. e.* 3^2 ratios between the even functions in the case of two arguments and 135, which is not a square, ratios between the even functions in the case of four arguments. In volume 94 of Crelle, pp. 74–86, F. Caspary, in a paper entitled *Zur Theorie der Thetafunctionen mit zwei Argumenten,* has obtained the above relation from a more general one; he shows that certain 16 pairs of products of double theta-functions arranged in the form of a determinant of the fourth order satisfy the necessary conditions of an orthogonal substitution. Caspary's fundamental theorem is as follows; using the notation which I have been employing: "If $x_1 x_2$; $y_1 y_2$ denote two pairs of independent arguments, then the sixteen theta products arranged in the following order:

$$\left\{\begin{matrix} \vartheta_{13}(x_1, x_2)\vartheta_{13}(y_1, y_2) & \vartheta_{3}(x_1, x_2)\vartheta_{3}(y_1, y_2) & -\vartheta_{10}(x_1, x_2)\vartheta_{10}(y_1, y_2) & -\vartheta_{8}(x_1, x_2)\vartheta_{8}(y_1, y_2) \\ -\vartheta_{7}(x_1, x_2)\vartheta_{7}(y_1, y_2) & \vartheta_{9}(x_1, x_2)\vartheta_{9}(y_1, y_2) & \vartheta_{4}(x_1, x_2)\vartheta_{4}(y_1, y_2) & -\vartheta_{14}(x_1, x_2)\vartheta_{14}(y_1, y_2) \\ \vartheta_{2}(x_1, x_2)\vartheta_{2}(y_1, y_2) & -\vartheta_{16}(x_1, x_2)\vartheta_{16}(y_1, y_2) & \vartheta_{5}(x_1, x_2)\vartheta_{5}(y_1, y_2) & -\vartheta_{11}(x_1, x_2)\vartheta_{11}(y_1, y_2) \\ \vartheta_{12}(x_1, x_2)\vartheta_{12}(y_1, y_2) & \vartheta_{6}(x_1, x_2)\vartheta_{6}(y_1, y_2) & \vartheta_{15}(x_1, x_2)\vartheta_{15}(y_1, y_2) & \vartheta_{1}(x_1, x_2)\vartheta_{1}(y_1, y_2) \end{matrix}\right\}$$

form the coefficients of a linear substitution in which the sum of the squares of the four new variables is equal to the sum of the squares of the four original variables multiplied by a certain factor." To convert the above arrangement into Prof. Cayley's "current number" notation, it is only necessary to diminish each subscript by unity. The notation adopted by Caspary is that employed by Weierstrass. The arguments x and y are connected with the arguments u and v by the relations

$$x_1 = u_1 + u'_1,\ y_1 = u_1 - u'_1$$

$$x_1 + y_1 = 2u_1,\ x_1 - y_1 = 2u'_1, \text{ etc.}$$

It seems to me that one ought to be able by suitably generalizing Caspary's theorem to find the corresponding relation for the quadruple functions. That would seem to be probably the most desirable way of going to work. Caspary passes from the above very general theorem to the particular case which I have cited above, viz. to the relations existing between the ratios of the zero-values of the even double theta-functions. A suitable generalization of his theorem would doubtless lead easily to the corresponding relation between the zero-values of the quadruple functions. This, however, I shall reserve for the second part of this paper.